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The Relativistic Scattering States of the Hulthén Potential with an Improved New Approximate Scheme to the Centrifugal Term

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Abstract The approximately analytical scattering state solutions of the l-wave Klein-Gordon equation with the unequal scalar and vector Hulthén potentials are carried out by an improved new approximate scheme to the centrifugal term. The normalized analytical radial wave functions of the l-wave Klein-Gordon equation with the mixed Hulthén potentials are presented and the corresponding calculation formula of phase shifts is derived. It is well shown that the energy levels of the continuum states reduce to those of the bound states at the poles of the scattering amplitude. Some useful figures are plotted to show the improved accuracy of our results and two special cases for s-wave ($l = 0$) and for $l = 0$ and equal scalar and vector Hulthén potentials are also studied briefly.

Keywords Hulthén potential · Scattering states · Klein-Gordon equation · Approximately analytical solutions

1 Introduction

There has been a growing interest in studying the exact solutions within the framework of non-relativistic quantum mechanics and relativistic quantum mechanics since they contain all the necessary information regarding the quantum system under consideration. For example, the exact solutions of the Schrödinger equation for a hydrogen atom and for a harmonic

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oscillator in three dimensions are an important milestone at the beginning stage of quantum mechanics, which provided a strong evidence for supporting the correctness of the quantum theory [1–3]. However, the exact solutions are few so that many quantum systems have to be treated by approximate methods. For example, some authors have approximately solved the Schrödinger and Klein-Gordon equations by a proper approximation [4–6] to the centrifugal term for some potential such as Manning-Rosen potential [7], the Eckart potential [8], the hyperbolic-type molecular potential [9], Hulthén potential [10–15], the Pöschl-Teller potential [16], etc. However, it should be noted that these contributions are mainly made to the study of the bound states than that of the scattering states for a given quantum system. Nevertheless, we have to study both of them in order to understand the studied quantum system completely. Based on the previous work [4–9], we have applied a proper approximation to the centrifugal term and obtained approximately analytical l-wave scattering solutions of the Schrödinger equation with the Manning-Rosen potential [17], the Eckart potential [18]. On the other hand, It should be noted that the results obtained previously [7–9] are in good agreement with those obtained by using the program based on a numerical integration procedure [19] for the short-range potential, but the difference between them appears for the long-range potential. Therefore, it is necessary and considerable interest to find a new approximation approach to deal with the quantum systems bound state and scattering state problems. In the present paper, we attempt to study the relativistic scattering states of arbitrary l-wave Klein-Gordon equation with the unequal scalar and vector Hulthén potentials by an improved new approximate scheme to the centrifugal term.

The Hulthén potential is one of the most important exponential model potential [20], which has been widely used in a number of areas such as nuclear and particle physics [21], atomic physics [22], molecular physics [23, 24], and chemical physics [25]. Its expression can be given by

$$V(r) = -\frac{V_0}{e^{r/r_0} - 1}, \tag{1}$$

where r_0 is related to the range of the potential, and V_0 can be seen as the depth of potential well. The reader can refer to [10, 26–32] for more information and possible applications of this potential.

The purpose of this work is two-fold. Firstly, we study the relativistic scattering states of the l-wave Klein-Gordon equation with unequal scalar and vector Hulthén potentials by an improved new approximate scheme to the centrifugal term, which has not been performed to our knowledge. Secondly, we try to show the relation between the energy levels of the continuum states and those of bound states at the poles of scattering amplitude. Undoubtedly, this study will provide a good reference to interpret theoretically the quantum system with the arbitrary l states.

This paper is organized as follows. In Sect. 2 we obtain approximately analytical scattering state solutions of the Klein-Gordon equation for the Hulthén potential with centrifugal term. Section 3 is devoted to briefly studying two special cases for s-wave and for $l = 0$ and equal scalar and vector Hulthén potentials. Some useful figures are plotted to show the improved accuracy of our results in Sect. 4. The conclusions are given finally in Sect. 5.

2 Scattering States of the Arbitrary l-Wave Klein-Gordon Equation

In spherical coordinates, the Klein-Gordon equation with the scalar potential $S(r)$ and vector potential $V(r)$ is given by ($\hbar = c = 1$)

$$\{-\nabla^2 + [M + S(r)]^2\}\Psi(r,\theta,\phi) = [E - V(r)]^2\Psi(r,\theta,\phi). \tag{2}$$

By taking $\Psi(r,\theta,\phi)=r^{-1}u(r)Y_{lm}(\theta,\phi)$, and inserting it into (2), we obtain the radial equation as

$$\frac{d^2u(r)}{dr^2}+\left\{[E^2-M^2]-2\big[MS(r)+EV(r)\big]+\big[V^2(r)-S^2(r)\big]-\frac{l(l+1)}{r^2}\right\}u(r)=0. \quad (3)$$

Here we consider the case that the scalar potential and vector potential are unequal the Hulthén potentials, i.e.,

$$V(r)=-\frac{V_0}{e^{r/r_0}-1},\qquad S(r)=-\frac{S_0}{e^{r/r_0}-1}. \quad (4)$$

The substitution of (4) into (3) gives

$$\frac{d^2u(r)}{dr^2}+\left\{\frac{\lambda^2}{r_0^2}+\frac{\beta^2/r_0^2}{e^{r/r_0}-1}-\frac{\alpha^2/r_0^2}{(e^{r/r_0}-1)^2}-\frac{l(l+1)}{r^2}\right\}u(r)=0, \quad (5)$$

where

$$\lambda=r_0k,\qquad k=\sqrt{E^2-M^2},\qquad \beta=r_0\sqrt{2EV_0+2MS_0},\qquad \alpha=r_0\sqrt{S_0^2-V_0^2}. \quad (6)$$

Obviously, (5) cannot be solved analytically due to the centrifugal term [33, 34]. Therefore, we must use a proper approximation to the centrifugal term similar to other authors. Unlike the following approximation used in the previous work [7–9, 17, 18],

$$\frac{1}{r^2}\approx\frac{e^{-r/r_0}}{r_0^2(1-e^{-r/r_0})^2}, \quad (7)$$

here we apply an improved new approximate scheme to the centrifugal term [35]

$$\frac{1}{r^2}\approx\frac{1}{r_0^2}\left[\frac{\omega e^{-r/r_0}}{1-e^{-r/r_0}}+\frac{e^{-2r/r_0}}{(1-e^{-r/r_0})^2}\right], \quad (8)$$

which reduces to (7) when $\omega=1$.

We now solve (5). Inserting (8) into (5) allows us to obtain

$$\frac{d^2u(r)}{dr^2}+\left\{\frac{\lambda^2}{r_0^2}+\frac{\beta^2/r_0^2}{e^{r/r_0}-1}-\frac{\alpha^2/r_0^2}{(e^{r/r_0}-1)^2}-\frac{l(l+1)}{r_0^2}\left[\frac{\omega e^{-r/r_0}}{1-e^{-r/r_0}}+\frac{e^{-2r/r_0}}{(1-e^{-r/r_0})^2}\right]\right\}u(r)=0. \quad (9)$$

Introducing a new variable $z=1-e^{-r/r_0}$, ($r\in[0,\infty]$, $z\in[0,1]$), (9) can be rearranged as

$$\frac{d^2u(z)}{dz^2}-\frac{1}{1-z}\frac{du(z)}{dz}+\left[\frac{\lambda^2}{(1-z)^2}+\frac{\beta^2}{z(1-z)}-\frac{\alpha^2}{z^2}-\frac{\omega l(l+1)}{z(1-z)}-\frac{l(l+1)}{z^2}\right]u(z)=0. \quad (10)$$

According to Frobenius theorem, the singularity points of the above differential (10) play an important role in the form of the wave functions. The singular points here are at $z=0$ and at $z=1$. As a result, we take wave function of the form

$$u(z)=z^{\delta}(1-z)^{-i\lambda}f(z), \quad (11)$$

where

$$\delta=\frac{1}{2}\big[1+\sqrt{(2l+1)^2+4\alpha^2}\big]=\frac{1}{2}\big[1+\sqrt{(2l+1)^2+4r_0^2(S_0^2-V_0^2)}\big]. \quad (12)$$

Inserting (11) into (10), we have

$$z(1-z)\frac{d^2 f(z)}{dz^2}+\left[2\delta-(2\delta+1-2i\lambda)z\right]\frac{df(z)}{dz}+\left[\beta^2+2i\lambda\delta-\delta-\omega l(l+1)\right]f(z)=0. \quad (13)$$

The general solutions of above equation are nothing but the hypergeometric functions [36], it can be given by

$$f(z)=C_{1\,2}F_1(a,b;c;z)+C_2 z^{1-c}{}_2F_1(a-c+1,b-c+1;2-c;z), \quad (14)$$

where

$$a=\delta-i\lambda+\xi, \qquad b=\delta-i\lambda-\xi, \qquad c=2\delta, \qquad \xi=\sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)}. \quad (15)$$

Considering the boundary condition i.e., $z\to 0$ ($r\to 0$), $u(z)$ tending to finite, the allowed solution is

$$u(z)=z^{\delta}(1-z)^{-i\lambda}{}_2F_1(a,b;c;z), \quad (16)$$

and the correspondingly analytical wave functions of arbitrary l-wave relativistic scattering states for the unequal scalar and vector Hulthén potentials can be written as

$$u(r)=N(1-e^{-r/r_0})^{\delta}e^{i\lambda r/r_0}{}_2F_1(a,b;c;1-e^{-r/r_0}), \quad (17)$$

where a, b and c are given by (15), and N is the normalized constant to be determined below.

Before deriving the phase shifts, let us recall a recurrence relation of hypergeometric function [36], which is used to analyze the asymptotic behavior of the wave function and present the normalized constant and phase shifts,

$$\begin{aligned}{}_2F_1(a;b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b-c+1;1-z)\\ &\quad+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}{}_2F_1(c-a,c-b;c-a-b+1;1-z),\end{aligned} \quad (18)$$

with which, and considering ${}_2F_1(a,b;c;0)=1$ for $r\to\infty$, we have

$${}_2F_1(a,b;c;1-e^{-r/\beta})\xrightarrow{r\to\infty}\Gamma(c)\left\{\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}+e^{-2ikr}\left[\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right]^*\right\}. \quad (19)$$

In above derivation, we have used the following relations,

$$\begin{aligned}c-a-b&=2i\lambda=(a+b-c)^*,\\ c-a&=\delta+i\lambda-\sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)}=b^*,\\ c-b&=\delta+i\lambda+\sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)}=a^*.\end{aligned} \quad (20)$$

By taking $\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}=|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}|e^{i\delta'}$, and inserting this into above equation (19), we have

$${}_2F_1(a,b;c;1-e^{-r/r_0})\xrightarrow{r\to\infty}\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|e^{-i\lambda r/r_0}[e^{i(\lambda r/r_0+\delta')}+e^{-i(\lambda r/r_0+\delta')}], \quad (21)$$

which is inserted into (17), we have the asymptotic form of the formula (17) for $r \to \infty$,

$$u(r) \stackrel{r\to\infty}{\longrightarrow} 2N\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\sin(kr+\pi/2+\delta'). \tag{22}$$

In the above derivation, we have used the relation of k and λ, i.e., (6). By comparing (22) with the boundary condition [2, 37] $r \to \infty \Rightarrow u(\infty) \to 2\sin(kr - \frac{1}{2}l\pi + \delta_l)$, the phase shifts and the normalized constant can be given by

$$\delta_l = \frac{\pi}{2} + \frac{1}{2}l\pi + \delta' = \frac{\pi}{2} + \frac{1}{2}l\pi + \arg\Gamma(c-a-b) - \arg\Gamma(c-a) - \arg\Gamma(c-b), \tag{23}$$

$$N = \frac{1}{\Gamma(c)}\left|\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c-a-b)}\right|. \tag{24}$$

The substitution of (20) into (23), (24) gives the phase shifts and the normalized constant as follows

$$\begin{aligned}\delta_l = \frac{\pi}{2} + \frac{1}{2}l\pi + \delta' = \frac{\pi}{2} + \frac{1}{2}l\pi &+ \arg\Gamma(2i\lambda)\\ &- \arg\Gamma\left(\delta + i\lambda - \sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)}\right)\\ &- \arg\Gamma(\delta + i\lambda + \sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)}),\end{aligned} \tag{25}$$

$$\begin{aligned}N = &\frac{|\Gamma(\delta + i\lambda - \sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)})|}{\Gamma(2\delta)}\\ &\times\left|\frac{\Gamma(\delta + i\lambda + \sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)})}{\Gamma(2i\lambda)}\right|.\end{aligned} \tag{26}$$

Before ending this part, let us study the properties of the scattering amplitude. From the general theory of the partial-wave method, the scattering amplitude is defined by

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)\left[\frac{e^{2i\delta_l}-1}{2ik}\right]P_l(\cos\theta), \tag{27}$$

where l is the angular quantum number. For this purpose, based on (25), we discuss the properties of $\Gamma(\delta + i\lambda + \sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)})$. From the definition of the Gamma function

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \cdots, \tag{28}$$

we know that $z = 0, -1, -2, \ldots$ are the first order poles of the $\Gamma(z)$, i.e. the first order poles of $\Gamma(\delta + i\lambda + \sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)})$ are situated at

$$\delta + i\lambda + \sqrt{\delta^2-\delta-\lambda^2+\beta^2-\omega l(l+1)} = 0, -1, -2, \ldots = -n_r \quad (n_r = 0, 1, 2, \ldots). \tag{29}$$

At the poles of the scattering amplitude, the corresponding energy equation are given by

$$-\lambda^2 = \left[\frac{\beta^2 - n^2 - \omega l(l+1) - (2n-1)(\delta-1)}{2(n+\delta-1)}\right]^2, \quad (n = 1, 2, 3, \ldots). \tag{30}$$

Comparing the above equation (30) with (18) of [10] and observing the relations of corresponding parameters, i.e., $-\lambda^2 \leftrightarrow \alpha^2$, $\delta \leftrightarrow \delta'$. It is not difficult to find that (30) is consistent with (18) in [10] when $\omega = 1$. It is well shown that the poles of S-matrix in the complex energy plane correspond to bound states for real poles and scattering states for complex poles in the lower half of the energy plane [38, 39]. That is to say, the energy levels of the continuum states reduce to those of the bound states at the poles of the scattering amplitude.

3 Discussion

In this section we study two special cases. Firstly, let us discuss the special case $l = 0$. For this case, from (12) and (15), we have

$$\delta = \frac{1}{2}\left[1 + \sqrt{1 + 4r_0^2(S_0^2 - V_0^2)}\right], \qquad a = \delta - i\lambda + \sqrt{\delta^2 - \delta - \lambda^2 + \beta^2},$$
$$b = \delta - i\lambda - \sqrt{\delta^2 - \delta - \lambda^2 + \beta^2}, \qquad c = 2\delta. \tag{31}$$

If so, the corresponding results reduce to those of the exact solutions of s-wave relativistic scattering state for the Hulthén potential

$$\delta_l = \frac{\pi}{2} + \delta' = \frac{\pi}{2} + \arg\Gamma(2i\lambda) - \arg\Gamma\left(\delta + i\lambda - \sqrt{\delta^2 - \delta - \lambda^2 + \beta^2}\right)$$
$$- \arg\Gamma\left(\delta + i\lambda + \sqrt{\delta^2 - \delta - \lambda^2 + \beta^2}\right), \tag{32}$$

$$N = \frac{|\Gamma(\delta + i\lambda - \sqrt{\delta^2 - \delta - \lambda^2 + \beta^2})|}{\Gamma(2\delta)} \times \left|\frac{\Gamma(\delta + i\lambda + \sqrt{\delta^2 - \delta - \lambda^2 + \beta^2})}{\Gamma(2i\lambda)}\right|, \tag{33}$$

$$u(r) = N(1 - e^{-r/r_0})^{\lambda} e^{i\lambda r/r_0} {}_2F_1(a, b; c; 1 - e^{-r/r_0}). \tag{34}$$

Second, we study the special case $l = 0$ and equal scalar and vector potentials ($S_0 = V_0$). In this case, we have $\delta = 1$ and $\Gamma(2\delta) = 1$, and the corresponding parameters are simplified as

$$a = 1 - i\lambda + \sqrt{\beta^2 - \lambda^2}, \qquad b = 1 - i\lambda - \sqrt{\beta^2 - \lambda^2}, \qquad c = 2, \tag{35}$$

and it is found that the corresponding equation (3) reduces to a Schrödinger-like equation with the potential $\tilde{V}(r) = -\frac{V_0(M+E)}{e^{r/r_0}-1}$ and energy $\tilde{E} = \frac{E^2-M^2}{2}$. The results given in (17), (25) and (26) reduce to

$$\delta_l = \frac{\pi}{2} + \delta' = \frac{\pi}{2} + \arg\Gamma(2i\lambda) - \arg\Gamma\left(1 + i\lambda - \sqrt{\beta^2 - \lambda^2}\right)$$
$$- \arg\Gamma\left(1 + i\lambda + \sqrt{\beta^2 - \lambda^2}\right), \tag{36}$$

$$N = \left|\frac{\Gamma(1 + i\lambda - \sqrt{\beta^2 - \lambda^2})\Gamma(1 + i\lambda + \sqrt{\beta^2 - \lambda^2})}{\Gamma(2i\lambda)}\right|, \tag{37}$$

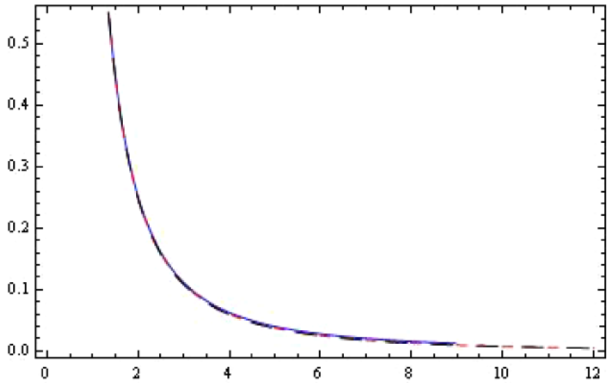
$$u(r) = N(1 - e^{-r/r_0})^{\delta} e^{i\lambda r/r_0} {}_2F_1(a, b; c; 1 - e^{-r/r_0}). \tag{38}$$

Comparing the above equations (36), (37) and (38) with (29), (30) and (31) of [18] and observing the corresponding relations of potential and energy, i.e., $\tilde{V}(r) = -\frac{V_0(M+E)}{e^{r/r_0}-1} \leftrightarrow V(r) = \frac{-\alpha e^{-r/a}}{1-e^{-r/a}}$ and $\tilde{E} = \frac{E^2-M^2}{2} \leftrightarrow E$. It is obvious to show that the above expressions (36)–(38) completely coincide with expressions (29)–(31) of exact solutions of s-wave scattering state for the Hulthén potential in [18].

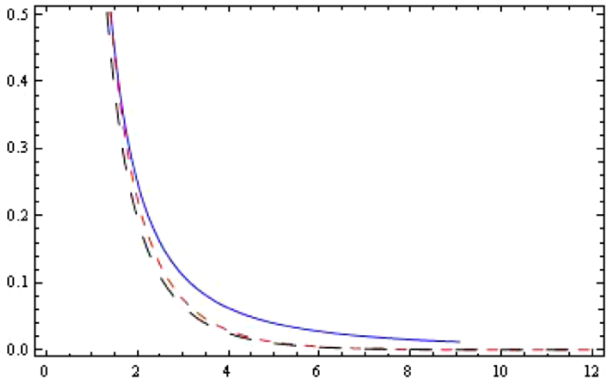
4 Accuracy Analysis

To show the improved accuracy of our results, The plots of the centrifugal term $1/r^2$ (blue solid line), the improved new approximation to it $1/r^2 \approx [\omega e^{-r/r_0}/(1-e^{-r/r_0}) + e^{-2r/r_0}/(1-e^{-r/r_0})^2]/r_0^2$ (red dashed) and the conventional approximation to it $1/r^2 \approx e^{-r/r_0}/[r_0^2(1-e^{-r/r_0})^2]$ (black long dashed) as the functions of the variable r are displayed from Figs. 1a–e with different potential range parameter $r_0 = 5.15, 1.15, 0.85, 0.75, 0.65$, and corresponding parameter $\omega = 1.018, 1.180, 1.380, 1.520, 1.700$. It is obvious to show that the conventional approximation (7) is a good approximation to the centrifugal term for short potential range, i.e., large r_0 (for example, $r_0 = 5.15, 1.15$), but not a good approximation to the centrifugal term for long potential range, i.e., small r_0 (for example, $r_0 = 0.85, 0.75, 0.65$). In order to improve the accuracy in studying the scattering states of the Hulthén potential in all potential range, the improved new approximate scheme (8) has been applied in the present work. It should be mentioned that parameter ω is an effectively adjustable parameter and should be very close to 1 for short potential range, i.e. large r_0, and bigger than 1 for long potential range, i.e., small r_0 as shown in Figs. 1a–e. Although the parameter ω could not make good adjustable function when variable $r > 4$ (this is because the centrifugal term and the approximations (7), (8) to it are all rapidly near to zero.), the im-

Fig. 1 (Color online) The centrifugal term $1/r^2$ (*blue solid line*), the improved new approximation to it $1/r^2 \approx [\omega e^{-r/r_0}/(1-e^{-r/r_0}) + e^{-2r/r_0}/(1-e^{-r/r_0})^2]/r_0^2$ (*red dashed*) and the conventional approximation to it $1/r^2 \approx e^{-r/r_0}/[r_0^2(1-e^{-r/r_0})^2]$ (*black long dashed*) as the functions of the variable r are displayed in **a**–**e** with different potential range parameters r_0 and corresponding parameter ω

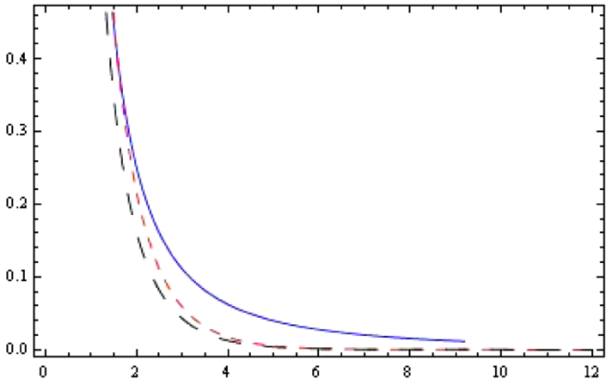


(a) $r_0 = 5.15, \omega = 1.018$

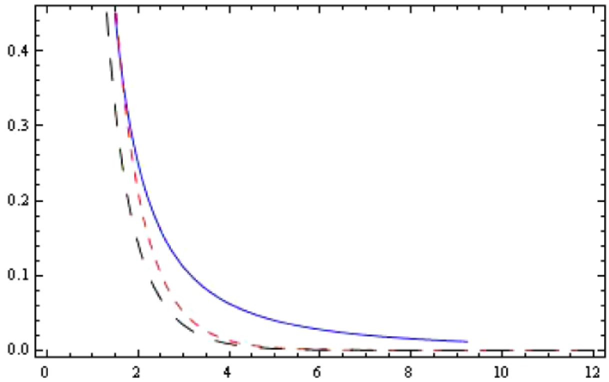


(b) $r_0 = 1.15, \omega = 1.180$

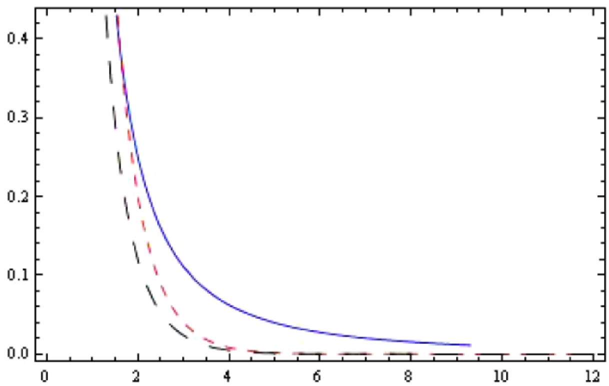
Fig. 1 (*Continued*)



(c) $r_0 = 0.85$, $\omega = 1.380$



(d) $r_0 = 0.75$, $\omega = 1.520$



(e) $r_0 = 0.65$, $\omega = 1.700$

proved new approximate scheme given in (8) is obviously much better than the conventional approximation given in (7) as shown in Figs. 1a–e, especially when variable $r < 4$.

5 Conclusion

In this work the approximately analytical scattering state solutions of the l-wave Klein-Gordon equation with the unequal scalar and vector Hulthén potentials have been presented by taking an improved new approximation to the centrifugal term. The normalized analytical radial wave functions of l-wave scattering state are obtained and the corresponding calculation formula of phase shifts is derived. It is shown that the energy levels of the continuum states reduce to those of the bound states at the poles of the scattering amplitude. Also, we have studied two special cases for $l = 0$ and for $l = 0$ and equal scalar and vector potentials. It is found that these obtained results reduce to exact solutions of s-wave scattering states of the Hulthén potential. We finally check the improved accuracy of our results.

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